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Integer moments of complex Wishart matrices and Hurwitz numbers

Fabio Deelan Cunden, Antoine Dahlqvist and Neil O’Connell ¹

Abstract. We give formulae for the cumulants of complex Wishart (LUE) and inverse Wishart matrices (inverse LUE). Their large- N expansions are generating functions of double (strictly and weakly) monotone Hurwitz numbers which count constrained factorisations in the symmetric group. The two expansions can be compared and combined with a duality relation proved in [F. D. Cunden, F. Mezzadri, N. O’Connell and N. J. Simm, *Commun. Math. Phys.* (2019)] to obtain: i) a combinatorial proof of the reflection formula between moments of LUE and inverse LUE at genus zero and, ii) a new functional relation between the generating functions of monotone and strictly monotone Hurwitz numbers. The main result resolves the integrality conjecture formulated in [F. D. Cunden, F. Mezzadri, N. J. Simm and P. Vivo, *J. Phys. A* 49 (2016)] on the time-delay cumulants in quantum chaotic transport. The precise combinatorial description of the cumulants given here may cast new light on the concordance between random matrix and semiclassical theories.

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1. Introduction and results

1.1. Time-delay matrix and an integrality conjecture. Random matrices have been used to model a variety of scattering phenomena in complex systems including heavy nuclei, disordered mesoscopic conductors, and chaotic quantum billiards. See, e.g., [2, 31, 62, 57]. The time-dependent aspects of a scattering process are usually described by the time-delay (or

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Wigner-Smith) matrix Q . Its eigenvalues τ_j are called *proper delay times* and can be thought of as the time spent by an incident wave in the scattering region at a propagating mode (or open channel) $j = 1, \dots, N$. See [60] for a modern introduction.

A statistical approach to the time-delay based on random matrices was developed in the 1990s, see [38, 25, 58, 27]. For ballistic quantum dots with perfect coupling (a physical realisation of chaotic quantum billiards), Brouwer, Frahm, and Beenakker [10, 11] argued that the inverses of the proper delay times $\lambda_j = (N\tau_j)^{-1}$ are distributed according to the Laguerre ensemble of random matrix theory

$$p(d\lambda_1, \dots, d\lambda_N) = c_{N,\beta} \prod_{i < j} |\lambda_i - \lambda_j|^\beta \prod_k \lambda_k^{\beta N/2} e^{-\beta N \lambda_k/2} \chi_{\mathbb{R}_+}(\lambda_k) d\lambda_k, \quad (1)$$

where $\beta \in \{1, 2, 4\}$ indicates orthogonal, unitary, or symplectic symmetry, respectively, and $c_{N,\beta}$ is a normalisation constant. This provided a route to apply various techniques from random matrix theory for the calculations of expectation values, typical fluctuations and tails of the distributions of the time-delay moments $\text{tr} Q^k$. See [42, 56, 3, 36, 61, 18, 20, 21, 52, 50, 45, 46, 47].

Notation. Tr denotes the non-normalised trace on $\mathcal{M}_N(\mathbb{C})$, and $\text{tr} = \frac{1}{N} \text{Tr}$. For $n \in \mathbb{N}$, we set $[n] = \{1, \dots, n\}$, and $\mathcal{P}(n)$ is the set of partitions of $[n]$. If (Y_1, \dots, Y_ℓ) are random variables (not necessarily distinct) on the same probability space with finite moments, their ℓ th cumulant (or connected average) is defined according to the formula $C_\ell(Y_1, \dots, Y_\ell) = \sum_{\pi \in \mathcal{P}(\ell)} (|\pi| - 1)! (-1)^{|\pi|-1} \prod_{B \in \pi} \mathbb{E} \prod_{i \in B} Y_i$.

The joint law (1) of the eigenvalues of $W = (NQ)^{-1}$ defines a β -ensemble ($\beta > 0$) with a strictly convex potential. This case belongs to the class of one-cut, off-critical ensembles, for which Borot and Guionnet [9] proved the existence of asymptotic $1/N$ -expansions determined by recursive relations known as ‘loop equations’. For instance, the generating series of the cumulants (also called ‘correlators’)

$$G_{\ell,\beta}(z_1, \dots, z_\ell) = C_\ell \left(\text{tr} \frac{1}{z_1 - W}, \dots, \text{tr} \frac{1}{z_\ell - W} \right) \quad (2)$$

admit large- N asymptotic expansions of the form

$$G_{\ell,\beta}(z_1, \dots, z_\ell) = \frac{1}{(\beta N^2)^{n-1}} \sum_{g \geq 0} N^{-g} G_{\ell,\beta}^{\{g\}}(z_1, \dots, z_\ell), \quad (3)$$

where $G_{\ell,\beta}^{\{g\}}$ has a very simple dependence in β

$$G_{\ell,\beta}^{\{g\}}(z_1, \dots, z_\ell) = \sum_{k=0}^{\lfloor g/2 \rfloor} \beta^{-k} \left(\frac{1}{2} - \frac{1}{\beta} \right)^{g-2k} G_{\ell}^{\{k;g-2k\}}(z_1, \dots, z_\ell). \quad (4)$$

(See [9] for details.) The coefficients $G_{\ell}^{\{k;g-2k\}}$ can be computed recursively using the Chekhov-Eynard topological recursion[12]. It is easy to check that, when $\beta = 2$, $G_{\ell,2}^{\{g\}} = 0$ if g is odd, and (3) is an expansion in powers of $1/N^2$.

In [20], using methods devised by Ambjørn, Chekhov, Kristjansen, and Makeenko[1], the explicit form of the leading order $G_{\ell,\beta}^{\{0\}}(z_1, \dots, z_\ell)$, and the large- N limit of the cumulants

$$\lim_{N \rightarrow \infty} (\beta N^2)^{\ell-1} C_{\ell}(\text{tr} W^{-\mu_1}, \dots, \text{tr} W^{-\mu_\ell}) = c_0(\mu_1, \dots, \mu_\ell) \quad (5)$$

were analysed. (The limit does not depend on β .) Extensive computations of some families of $c_0(\mu_1, \dots, \mu_\ell)$'s led the authors to the following integrality conjecture.

Conjecture 1 ([20]). *For all $\ell \geq 1$ and $(\mu_1, \dots, \mu_\ell) \in \mathbb{N}^\ell$,*

$$c_0(\mu_1, \dots, \mu_\ell) \in \mathbb{N}.$$

The present work started as an attempt to prove the conjecture. In this paper we provide an explicit formula for the $1/N$ -expansion of the cumulants

$$C_{\ell}(\text{tr} W^{-\mu_1}, \dots, \text{tr} W^{-\mu_\ell}) = \frac{1}{(2N^2)^{\ell-1}} \sum_{g \geq 0} N^{-g} c_g(\mu_1, \dots, \mu_\ell) \quad (6)$$

when $\beta = 2$. The result not only resolves Conjecture 1, but shows that *the full $1/N$ -expansion has positive integer coefficients* (i.e., $c_g(\mu_1, \dots, \mu_\ell) \in \mathbb{N}$) whose combinatorial interpretation we describe completely in terms of constrained factorisations in the symmetric group. In fact, the large- N asymptotics (6) is a ‘genus’ expansion.

1.2. Complex Wishart matrices and the Laguerre unitary ensemble.

For any real number $M > N - 1$, consider the following probability measure supported on the cone of positive definite $N \times N$ complex Hermitian matrices

$$\gamma(dX) = \frac{N^{NM}}{\pi^{N(N-1)/2} \prod_{j=0}^{N-1} \Gamma(M-j)} (\det X)^{M-N} \exp(-N \text{Tr} X) dX. \quad (7)$$

A random matrix W distributed according to the above measure is a *complex Wishart matrix* with parameter M . It is also quite common to use the parameters $c = M/N$, or $\alpha = M - N$. The eigenvalues of W (we drop the dependence on N and c for notational convenience) are distributed according to

$$p(d\lambda_1, \dots, d\lambda_N) = c_N \prod_{i < j} |\lambda_i - \lambda_j|^2 \prod_k \lambda_k^{M-N} e^{-N\lambda_k} \chi_{\mathbb{R}_+}(\lambda_k) d\lambda_k$$

$$c_N^{-1} = \frac{N!}{N^{MN}} \prod_{j=1}^N \Gamma(\alpha + j) \Gamma(j).$$

This is the Laguerre Unitary Ensemble (LUE for short) of random matrix theory. When M is an integer, there is the equality in law $W = N^{-1} X X^\dagger$, where X is a $N \times M$ random matrix with independent standard Gaussian entries [48]. When $\beta = 2$, Eq. (1) is of this type for the particular choice $M = 2N$ (or $c = 2$, $\alpha = N$).

1.3. Statement of results.

Notation. When σ is a permutation, an integer partition or a set partition, we denote by $\#\sigma$ its number of cycles (resp. blocks). For a random matrix X of size N , with coefficients having joint moments of homogeneity $n \in \mathbb{N}$, we shall denote for any integer partition $\mu = (\mu_1, \dots, \mu_\ell) \vdash n$, the scaled cumulant

$$\mathcal{C}_X(\mu) = \frac{|\mu|!}{z_\mu} N^{2(\#\mu-1)} C_{\#\mu}(\text{tr}(X^{\mu_1}), \dots, \text{tr}(X^{\mu_\ell})), \quad (8)$$

where $|\mu| = n$, $\#\mu = \ell$ and $z_\mu = \prod_{i \geq 1} m_i! i^{m_i}$ (m_i being the number of parts of μ equal to i).

The main purpose of this paper is to explain that, for the LUE and inverse LUE, (8) counts combinatorial quantities, related to factorisations in the symmetric group.

Theorem 1. Fix $n \in \mathbb{N}$, $n \geq 1$, and $\mu \vdash n$. Then,

$$\mathcal{C}_{W^{-1}}(\mu) = \sum_{g \geq 0} N^{-2g} \sum_{\nu \vdash n} (c-1)^{-(n+2g-2+\#\mu+\#\nu)} H_g^\uparrow(\mu, \nu) \quad \text{for } c > 1 + \frac{n}{N}, \quad (9)$$

$$\mathcal{C}_W(\mu) = \sum_{g \geq 0} N^{-2g} \sum_{\nu \vdash n} c^{n-(2g-2+\#\mu+\#\nu)} H_g^{\uparrow\uparrow}(\mu, \nu) \quad \text{for } c > 1 - \frac{1}{N}. \quad (10)$$

$H_g^\uparrow(\mu, \nu)$ is the number of tuples $(\alpha, \tau_1, \dots, \tau_r, \beta)$, where

- (i) $r = \#\mu + \#\nu + 2g - 2$;
- (ii) $\alpha, \beta \in S_n$ are respectively permutations of type μ and ν and τ_1, \dots, τ_r are transpositions such that

$$\alpha \tau_1 \dots \tau_r = \beta;$$

- (iii) the group generated by $(\alpha, \tau_1, \dots, \tau_r)$ acts transitively on $[n]$;

- (iv) τ_1, \dots, τ_r being written as $\tau_i = (a_i b_i)$ with $a_i < b_i$,

$$b_1 \leq b_2 \leq \dots \leq b_r.$$

$H_g^{\uparrow\uparrow}(\mu, \nu)$ is the number of tuples $(\alpha, \tau_1, \dots, \tau_r, \beta)$, satisfying all the four conditions above but the last one, which is replaced by

- (iv') τ_1, \dots, τ_r being written as $\tau_i = (a_i b_i)$ with $a_i < b_i$,

$$b_1 < b_2 < \dots < b_r.$$

Note that the strict monotonicity condition (iv') truncates the sum in g , and $\mathcal{C}_W(\mu)$ is a *polynomial* in $1/N^2$ (this is well known). The series representation (9) of the cumulants $\mathcal{C}_{W^{-1}}(\mu)$ is not asymptotic but *convergent* for $N > n/(c-1)$.

The fact that $\mathcal{C}_{W^{-1}}(\mu)$ and $\mathcal{C}_W(\mu)$ can be written as sums over permutations is a consequence of the *Schur-Weyl duality*, which applies to *any* unitarily invariant ensemble. Explicit formulae for the coefficients in the sum are only known for special cases, e.g., GUE, CUE, and LUE. In fact, the expression (10) is folklore in the literature [17, 34, 39]. The new result here is the explicit formula (9) which shows that the class of ‘solvable’ matrix ensembles includes the inverse LUE too.

The numbers $H_g^\uparrow(\mu, \nu)$ (resp. $H_g^{\uparrow\uparrow}(\mu, \nu)$) in the above Theorem are known as *monotone* (resp. *strictly monotone*) *double Hurwitz numbers*, a

special class of *Hurwitz numbers*. The latter count factorisations without the condition (iv) or (iv') which are in bijection with labeled connected ramified coverings of the sphere of degree n , with ramifications of type μ, ν and r simple ramifications, with a total space defining a surface of genus g . For a beautiful introduction see [37]. The above statement can also be reformulated in terms of *prefixes of minimal factorisations*, see Theorem 2 below and, when μ has one block, in terms of *parking functions*, see [41, 7, 59]. When $\#\mu = n$, $H_g^\uparrow((1, \dots, 1), \nu)$ is the number of *primitive factorisations* of any permutation of cycle type ν into r transpositions, see [43, 26].

The main ingredients of the proof hinges on the combination of two results: i) a formula for the expectation of coefficients of inverse Wishart matrices found by Graczyk, Letac, and Massam [30] (in its reformulation in terms of Weingarten function due to Collins, Matsumoto, and Saad [16]), and ii) the expression of the Weingarten function in terms of Jucys-Murphy elements [35] due to Novak [54]. The paper [49] by Gupta and Nagar contains some hints on the existence of explicit formulae for the cumulants of the inverse LUE, and was instrumental in our study.

First expressions for asymptotics of the Weingarten function were examined in [13] using representation theory and then developed in [17, 15], to study scaled cumulants of unitary invariant matrix ensembles, in terms of the poset of partitioned permutations. The introduction of monotone Hurwitz numbers for the study of the Harisch-Chandra-Itzykson-Zuber integrals and unitary invariant matrix models was initiated in [54, 44, 29], see also [32, 22, 8] for recent studies of these observables thanks to topological recursion.

An immediate application of the main Theorem for $c = 2$ is the following corollary on the time-delay matrix.

Corollary 1. *When $\beta = 2$ (unitary symmetry), the large- N expansions (6) of the cumulants of the time-delay matrix have positive integer coefficients. More precisely, $c_{2g+1} = 0$, and*

$$c_{2g}(\mu_1, \dots, \mu_\ell) = 2^{\ell-1} \frac{z_\mu}{|\mu|!} \sum_{\nu \vdash |\mu|} H_g^\uparrow(\mu, \nu) \in \mathbb{N}. \quad (11)$$

(This implies, in particular, Conjecture 1.)

Example 1. *Let $n = 3$, $\mu = (1, 1, 1)$, and $g = 0$. We outline the calculations of $H_0^\uparrow((1, 1, 1), \nu)$ and $H_0^{\uparrow\uparrow}((1, 1, 1), \nu)$. The integer partitions $\nu \vdash n$ are $\nu = (3), (2, 1)$, and $(1, 1, 1)$.*

$\nu = (3)$: There are $\binom{3}{2}^2 = 9$ products of $r = 2$ transpositions in S_3 :

$$\begin{array}{ccc} (12)(13) & (12)(23) & (23)(12) \\ (23)(13) & (13)(23) & (13)(12) \\ (12)(12) & (13)(13) & (23)(23) \end{array}$$

6 of them are transitive (the first two rows in the table above) and produce a cycle type $\mu = (3)$, but only the 4 products in the upper-left corner are monotone, so $H_0^\uparrow((1, 1, 1), (3)) = 4$. The number of strictly monotone products is $H_0^{\uparrow\uparrow}((1, 1, 1), (3)) = 2$.

$\nu = (2, 1)$: There are $\binom{3}{2}^3 = 27$ products of $r = 3$ transpositions, and 24 of them are transitive and produce a cycle type $\nu = (2, 1)$. Only 12 products are monotone

$$\begin{array}{cccc} (12)(12)(13) & (12)(13)(23) & (13)(13)(23) & (23)(13)(13) \\ (12)(12)(23) & (12)(23)(13) & (13)(23)(13) & (23)(13)(23) \\ (12)(13)(23) & (12)(23)(23) & (13)(23)(23) & (23)(23)(13) \end{array}$$

so $H_0^\uparrow((1, 1, 1), (2, 1)) = 12$, but none of them is strictly monotone, so $H_0^{\uparrow\uparrow}((1, 1, 1), (2, 1)) = 0$.

$\nu = (1, 1, 1)$: Among the $\binom{3}{2}^4 = 81$ products of $r = 4$ transpositions, only 8 of them are transitive, produce a cycle type $\nu = (1, 1, 1)$ and are monotone, so $H_0^\uparrow((1, 1, 1), (1, 1, 1)) = 8$,

$$\begin{array}{cc} (12)(12)(13)(13) & (12)(12)(23)(23) \\ (12)(13)(23)(13) & (12)(23)(13)(23) \\ (23)(23)(13)(13) & (13)(13)(23)(23) \\ (13)(23)(13)(23) & (23)(13)(23)(13) \end{array}$$

There are no strictly monotone products, so $H_0^{\uparrow\uparrow}((1, 1, 1), (1, 1, 1)) = 0$.

From the above calculations we can conclude that $(\frac{z_{(1,1,1)}}{|(1,1,1)|!} = 1)$

$$\begin{aligned} \mathcal{C}_3(\text{tr}W^{-1}, \text{tr}W^{-1}, \text{tr}W^{-1}) &= \frac{1}{N^4} \left(\sum_{\nu \vdash 3} \frac{H_0^\uparrow((1, 1, 1), \nu)}{(c-1)^{4+\#\nu}} + O(N^{-2}) \right) \\ &= \frac{1}{N^4} \left(\frac{4}{(c-1)^5} + \frac{12}{(c-1)^6} + \frac{8}{(c-1)^7} + O(N^{-2}) \right) \\ \mathcal{C}_3(\text{tr}W^1, \text{tr}W^1, \text{tr}W^1) &= \frac{1}{N^4} \left(\sum_{\nu \vdash 3} c^{2-\#\nu} H_0^{\uparrow\uparrow}((1, 1, 1), \nu) + O(N^{-2}) \right) \\ &= \frac{1}{N^4} (2c + O(N^{-2})). \end{aligned}$$

These agree with known results [47, 20].

Example 2. We compute $\mathbb{E}\mathrm{tr}W^{-1}$ and $\mathbb{E}\mathrm{tr}W$. These cases correspond to the one-block $\mu = (1)$. From the formulae (9)-(10), we have

$$\begin{aligned}\mathbb{E}\mathrm{tr}W^{-1} &= \frac{z_{(1)}}{|(1)|!} \frac{1}{N^{2(\#(1)-1)}} \mathcal{C}_{W^{-1}}((1)) = \sum_{g \geq 0} N^{-2g} \sum_{\nu \vdash 1} (c-1)^{-1-2g} H_g^\uparrow((1), \nu) \\ \mathbb{E}\mathrm{tr}W &= \frac{z_{(1)}}{|(1)|!} \frac{1}{N^{2(\#(1)-1)}} \mathcal{C}_W((1)) = \sum_{g \geq 0} N^{-2g} \sum_{\nu \vdash 1} c^{1-2g} H_g^{\uparrow\uparrow}((1), \nu)\end{aligned}$$

Using $H_g^\uparrow((1), (1)) = H_g^{\uparrow\uparrow}((1), (1)) = \delta_{g0}$, we recover the well-known results [30, 49]

$$\mathbb{E}\mathrm{tr}W^{-1} = (c-1)^{-1}, \quad \mathbb{E}\mathrm{tr}W = c.$$

Example 3. Set $c = 2$ (or $\alpha = N$). We want to compute the second moment of the time-delay matrix $\mathbb{E}\mathrm{tr}W^{-2}$, corresponding to the one-block partition $\mu = (2)$. From the definition (8) and formula (9),

$$\mathbb{E}\mathrm{tr}W^{-2} = \frac{z_{(2)}}{|(2)|!} \frac{1}{N^{2(\#(2)-1)}} \mathcal{C}_{W^{-1}}((2)) = \sum_{g \geq 0} N^{-2g} \sum_{\nu \vdash 2} H_g^\uparrow((2), \nu),$$

where we used $n = |(2)| = 2$, $\ell = \#(2) = 1$, and $z_{(2)} = 2$. There are two possibilities: $\nu = (2)$ and $\nu = (1, 1)$. Therefore we must count the monotone solutions of the factorisation problems (ii) in S_2

$$\begin{cases} (1\ 2)\tau_1 \dots \tau_{2g} = (1\ 2) & \text{if } \nu = (2) \\ (1\ 2)\tau_1 \dots \tau_{2g+1} = \mathrm{id} & \text{if } \nu = (1, 1) \end{cases}$$

where we used condition (i). In S_2 there is only one transposition, $\tau = (1\ 2)$. Therefore in both cases there is only one path of the form $(1\ 2) \dots (1\ 2)$ (with $2g$ factors if $\nu = (2)$, and $2g + 1$ factors if $\nu = (1, 1)$), and this path is also connected and monotone (conditions (iii) and (iv)). Hence, $H_g^\uparrow((2), (2)) = H_g^\uparrow((2), (1, 1)) = 1$. Substituting in the formula, we get

$$\mathbb{E}\mathrm{tr}W^{-2} = \sum_{g \geq 0} N^{-2g} \sum_{\nu \in \{(2), (1, 1)\}} 1 = \frac{2}{1 - N^{-2}} = \frac{2N^2}{N^2 - 1},$$

in agreement with the known result [21, Appendix A]. Note that $\mathbb{E}\mathrm{tr}W^1 = 2$ (see Example 2); c.f. the reciprocity formula (16) below.

Remark 1 (Physical significance of Theorem 1). *The random matrix theory approach to quantum chaos is believed to be equivalent to perturbative calculations based on semiclassical considerations. In the time-delay problem, the random matrix averages correspond to sums over **pairs of correlated classical trajectories** connecting the leads (asymptotic waves) with the interior of the cavity (the scattering region). In fact, some hints in the formulation of Conjecture 1 came from the observation that the semiclassical calculations boil down to weighted enumeration of **diagrams** recording only the topology of the trajectories.*

The concordance between random matrix and semiclassical theories in open systems has been established recently by Berkolaiko and Kuipers [4, 5, 6] and Novaes [51] in the case of quantum transport (when the relevant matrix model is the CUE). They put the diagrammatic method of the semiclassical approximation on a rigorous footing, and recast the semiclassical evaluation of moments as a summation over factorisations of given permutations (implying that the contribution of a diagram is given by the unitary, or orthogonal, Weingarten function).

On the other hand, for the time-delay, the agreement between semiclassics and random matrices remains limited to the first eight moments [53], and to the leading and several subleading orders in the $1/N$ -expansion [36]. By Theorem 1, the coefficients in the $1/N$ -expansion of the time-delay are positive integers, thus supporting the equivalence with the semiclassical diagrammatic rules. Moreover, (11) provides an explicit formula for the cumulants as a sum over monotone factorisations of permutations (which are related to the Weingarten function). It may not be too much to hope that this result will stimulate further study of the semiclassical diagrams in the time-delay problem to establish the equivalence with random matrices to all orders in $1/N$.

In the proof we shall first get a less symmetric version of Theorem 1.

Theorem 2. *For any permutation $\alpha \in S_n$ with cycle type $\mu = (\mu_1, \dots, \mu_\ell) \vdash n$,*

$$N^{2(\ell-1)} C_\ell(\text{tr}(W^{-\mu_1}), \dots, \text{tr}(W^{-\mu_\ell})) = \sum_{r,d \geq 0} N^{-2d} (c-1)^{-n-r} \# \mathcal{F}_{n,r,d}^\uparrow(\alpha), \quad (12)$$

and

$$N^{2(\ell-1)} C_\ell(\text{tr}(W^{\mu_1}), \dots, \text{tr}(W^{\mu_\ell})) = \sum_{r,d \geq 0} N^{-2d} c^{n-r} \# \mathcal{F}_{n,r,d}^{\uparrow\uparrow}(\alpha) \quad (13)$$

where $\mathcal{F}_{n,r,d}^\uparrow(\alpha)$ (resp. $\mathcal{F}_{n,r,d}^{\uparrow\uparrow}(\alpha)$) is the set of transpositions tuples (τ_1, \dots, τ_r) where $\tau_i = (a_i b_i)$ with $a_i < b_i$ for all i , such that

- (1) $\#\alpha\tau_1 \dots \tau_r = \#\alpha + r - 2d$,
- (2) $\langle \alpha, \tau_1, \dots, \tau_r \rangle$ acts transitively on $[n]$,
- (3) $b_1 \leq b_2 \leq \dots \leq b_r$ (resp. $b_1 < b_2 < \dots < b_r$).

Remark 2. Within the Cayley graph on S_n generated by all transpositions, the distance between two permutations α and β is $d(\alpha, \beta) = |\alpha^{-1}\beta|$, where for any $\sigma \in S_n$, $|\sigma| = n - \#\sigma$. Any element (τ_1, \dots, τ_r) of $\mathcal{F}_{n,r,d}^\uparrow$ and $\mathcal{F}_{n,r,d}^{\uparrow\uparrow}$ defines a path in S_n with r steps that starts at α and ends at $\beta = \alpha\tau_1 \dots \tau_r$, with $d(\alpha, \beta) = r - 2d$. The number d quantifies the **defect** of the path from being a geodesic. The number of paths with fixed defect (without the condition of transitivity and monotonicity) were considered in [40].

An equivalent representation of the LUE cumulants $\mathcal{C}_W(\mu)$ is the following.

Proposition 1. For $c \geq 1, n \in \mathbb{N}^*$, and $\mu \vdash n$,

$$\mathcal{C}_W(\mu) = \sum_{\nu \vdash n, g \geq 0} N^{-2g} c^{n-(2g-2+\#\mu+\#\nu)} \mathcal{C}_g(\mu, \nu), \quad (14)$$

where $\mathcal{C}_g(\mu, \nu)$ denotes the number of pairs $(\alpha, \beta) \in S_n^2$, such that

- (1) $[\alpha] = \mu$ and $[\alpha.\beta] = \nu$
- (2) $\#\mu + \#\beta + \#\nu - n = 2 - 2g$
- (3) the group generated by α and β acts transitively on $[n]$.

The triple $(\alpha, \beta, (\alpha.\beta)^{-1})$ is called a *constellation* of genus g , see [37, Section 1.2.4].

When $\ell = 1$ and $N \rightarrow \infty$, Theorem 1 allows to prove the following duality.

Corollary 2. For $c > 1$,

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E} \text{tr} W^{-(n+1)}}{(c-1)^{-(n+1)}} = \lim_{N \rightarrow \infty} \frac{\mathbb{E} \text{tr} W^n}{(c-1)^n}. \quad (15)$$

This result can be obtained using analytic methods [18, 24, 19]. We give here a combinatorial proof relying on a relation between monotone and strictly monotone Hurwitz paths.

The duality (15) is the projection to leading order in $1/N$ of an exact reciprocity law for the LUE recently found in [19, Proposition 2.1]:

$$\mathbb{E} \text{tr} (NW)^{-(n+1)} = \left(\prod_{j=-n}^n \frac{1}{\alpha + j} \right) \mathbb{E} \text{tr} (NW)^n. \quad (16)$$

In the notation of this paper the above relation reads

$$N^{-(n+1)} \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)} \mathcal{C}_{W^{-1}}((n + 1)) = N^n \frac{\Gamma(\alpha - n)}{\Gamma(n)} \mathcal{C}_W((n)). \quad (17)$$

By Theorem 1, it is possible to rephrase the duality (16) (or (17)) as a functional relation between generating functions of monotone and strictly monotone Hurwitz numbers. Define the formal power series

$$H_g^\uparrow(n; x) = \sum_{\nu \vdash n} x^{-\#\nu} H_g^\uparrow((n), \nu), \quad (18)$$

$$H_g^{\uparrow\uparrow}(n; x) = \sum_{\nu \vdash n} x^{-\#\nu} H_g^{\uparrow\uparrow}((n), \nu). \quad (19)$$

Then, combining the duality (16) with the explicit formulae (9)-(10) for $\mathcal{C}_{W^{-1}}$ and \mathcal{C}_W , and comparing the coefficients of the $1/N$ -expansions we can get a functional relation for the generating functions (18) and (19). Note that

$$\prod_{j=-n}^n \frac{1}{\alpha + j} = \frac{1}{\alpha^{2n+1}} \prod_{j=1}^n \left(1 - \frac{j^2}{\alpha^2} \right) = \sum_{g \geq 0} h_g(1^2, \dots, n^2) \alpha^{-g} \quad (20)$$

where

$$h_g(1^2, \dots, n^2) = \sum_{\substack{\ell_1, \dots, \ell_n \geq 0 \\ \ell_1 + \dots + \ell_n = g}} 1^{2\ell_1} 2^{2\ell_2} \dots n^{2\ell_n}$$

is the complete symmetric function of degree g evaluated on the square integers $1^2, \dots, n^2$ (see Lemma 2 below). We learned from [43] that the numbers

$$T(n + g, n) = h_g(1^2, \dots, n^2) \quad (21)$$

are known as *Carlitz-Riordan central factorial numbers*, and are given by the explicit formula

$$T(a, b) = 2 \sum_{j=0}^n (-1)^{b-j} \frac{j^{2a}}{(b-j)!(b+j)!}. \quad (22)$$

Putting all together we get the following functional equation.

Proposition 2.

$$\left(\frac{x-1}{x}\right)^{n+1} H_g^\uparrow(n+1; x-1) = n \sum_{j=0}^g \left(\frac{x-1}{x}\right)^{2j} T(n+g-j, n) H_j^{\uparrow\uparrow}(n; x), \quad (23)$$

Functional relations and some explicit formulae for the generating functions of (monotone) Hurwitz numbers have been considered in the literature, see [29, 28, 23]. To our knowledge, the relation (23) is new. It would be interesting to find a combinatorial proof of it.

There exists a duality similar to (15), for covariances ($\ell = 2$) of LUE moments at leading order in $1/N$. If $\mu = (\mu_1, \mu_2) \vdash n$, then [19, Theorem 7.3]

$$\lim_{N \rightarrow \infty} \frac{\mathcal{C}_{W^{-1}}(\mu)}{(c-1)^{-|\mu|}} = \lim_{N \rightarrow \infty} \frac{\mathcal{C}_W(\mu)}{(c-1)^{|\mu|}}. \quad (24)$$

By Theorem 2, this is equivalent to a relation between generating functions

$$\sum_{r \geq 0} z^r \# \mathcal{F}_{n,r,0}^\uparrow(\alpha) = \sum_{r \geq 0} (z+1)^{n-r} \# \mathcal{F}_{n,r,0}^{\uparrow\uparrow}(\alpha) \quad \text{for } z > 0, \quad (25)$$

when $\alpha \in S_n$ has two cycles $\#[\alpha] = 2$.

The enumerative properties of the integer moments of Wishart matrices, suggest to reinterpret various known results in random matrix theory from a combinatorial point of view. It is known that the moments of LUE (and any other β -ensemble) satisfy a set of recursions known as ‘loop equations’ (see [19, Lemma 7.1]) and it is natural to expect that they have a combinatorial explanation.

A special property of the LUE, is its connection to the Laguerre polynomials which led Haagerup and Thorbjørnsen to discover an exact three-term recursive relation [33, Theorem 8.2] for moments of W (the analogue of the Harer-Zagier recursion of the GUE). Later, it was observed in [21] that the Haagerup-Thorbjørnsen recursion extends to the moments of W^{-1} . For the inverse LUE with parameter $c = 2$, the recursion reads

$$(N^2 - n^2)(n+1) \mathbb{E} \text{tr} W^{-(n+1)} - 3N^2(2n-1) \mathbb{E} \text{tr} W^{-n} + N^2(n-2) \mathbb{E} \text{tr} W^{-(n-1)} = 0.$$

Denote by $S(n, d) = \sum_{r \geq 0} \# \mathcal{F}_{n,r,d}^\uparrow((1 \dots n))$ the number of monotone paths in the Cayley graph on S_n that start at the full cycle $(1 \dots n)$ and, after an arbitrary (finite) number of steps τ_1, \dots, τ_r have a defect $2d$. Then, Theorem 2 combined with the three-term recursion above gives a recurrence

for the numbers $S(n, d)$:

$$(n+1)S(n+1, d+1) - 3(2n-1)S(n, d+1) + (n-2)S(n-1, d+1) = n^2(n+1)S(n+1, d) \quad (26)$$

The above recursion appeared in the random matrix approach to the time-delay [21, Corollary 1.4] where the initial conditions are

$$S(n, 0) = {}_2F_1 \left(\begin{matrix} 1-n, n \\ 2 \end{matrix}; -1 \right), \quad S(0, d) = \delta_{0,d}, \quad S(1, d) = \delta_{0,d}. \quad (27)$$

Note that $S(n, 0)$ is the large Schröder number.

The existing proofs of (17)-(24) and (26) are based on special properties of the Laguerre polynomials, but it should be possible to prove these remarkable formulae using algebraic methods. Further study is in progress.

2. Proofs

2.1. Proof of the main Theorem. We shall give a proof that hinges on the following two propositions. The first one is a restatement of [30, Theorems 1 and 4] and [16, Theorems 3.1 and 4.3] in a notation which is shorter and better adapted to the purposes of this paper.

Proposition 3 ([30, 16]). *For any $i, j \in [N]^n$*

$$\mathbb{E} \prod_{k=1}^n W_{i(k)j(k)} = N^{-n} \sum_{\substack{\sigma \in S_n: \\ i \circ \sigma = j}} \Omega_{n, cN}(\sigma),$$

and, for $c > 1 + \frac{n}{N}$,

$$\mathbb{E} \prod_{k=1}^n W_{i(k)j(k)}^{-1} = (-N)^n \sum_{\substack{\sigma \in S_n: \\ i \circ \sigma = j}} \Omega_{n, (1-c)N}^{-1}(\sigma),$$

where for any permutation $\sigma \in S_n$ and $z \in \mathbb{C}$,

$$\Omega_{n,z}(\sigma) = z^{\#\sigma},$$

whereas for $|z| > n-1$, $\Omega_{n,z}^{-1} : S_n \rightarrow \mathbb{C}$, denotes the unique function such that

$$\Omega_{n,z}^{-1} * \Omega_{n,z} = \Omega_{n,z} * \Omega_{n,z}^{-1} = \delta_{\text{id}},$$

where $*$ is the convolution product of functions on the symmetric group S_n .

The function $\Omega_{n,z}^{-1}$, more commonly denoted by $\text{Wg}_{n,z}$, is called the *unitary Weingarten function* and admits a remarkable factorisation property (Proposition 4 below). To state it, we shall identify the unital algebra $(\mathbb{C}^{S_n}, *, \delta_{\text{id}})$ with the group algebra $(\mathbb{C}[S_n], \cdot, \text{id})$, that is, the algebra of formal linear combinations of permutations with a product rule extending linearly the product of the group S_n , thanks to the isomorphism that maps a function $f \in \mathbb{C}^{S_n}$ to $\sum_{\sigma \in S_n} f(\sigma)\sigma \in \mathbb{C}[S_n]$. We shall keep abusively the same notations for $\Omega_{n,z}$ and $\Omega_{n,z}^{-1}$ viewed as elements of $\mathbb{C}[S_n]$ instead of functions.

The *Jucys-Murphy element* J_i [35] in $\mathbb{C}[S_n]$ is the sum of all transpositions interchanging i with a smaller number:

$$\begin{aligned} J_1 &= 0 \\ J_2 &= (1\ 2) \\ J_3 &= (1\ 3) + (2\ 3) \\ &\vdots \\ J_n &= (1\ n) + (2\ n) + \dots + (n-1\ n). \end{aligned}$$

They form a commutative family in the group algebra $\mathbb{C}[S_n]$.

Proposition 4 ([14, 54]). *For any $z \in \mathbb{C}$,*

$$\Omega_{n,z} = (z + J_1)(z + J_2) \cdots (z + J_n)$$

and, for any $z \in \mathbb{C} \setminus \{1 - n, 2 - n, \dots, n - 2, n - 1\}$,

$$\Omega_{n,z}^{-1} = (z + J_1)^{-1}(z + J_2)^{-1} \cdots (z + J_n)^{-1}.$$

In the proof of the main theorem we will use classical manipulation of cumulants.

Notation. Setting for partitions $\pi, \nu \in \mathcal{P}(n)$, $\pi \leq \nu$, whenever all blocks of π are included in those of ν , (the partition ν is said to be coarser than π) defines a structure of poset on $\mathcal{P}(n)$ with maximal element $1_n = \{[n]\}$ and minimal element $0_n = \{\{1\}, \{2\}, \dots, \{n\}\}$. For any $\mu \in \mathcal{P}(n)$, we shall write $\mathcal{P}(n)_{\geq \mu} = \{\pi \in \mathcal{P}(n) : \pi \geq \mu\}$ the set of partitions coarser than μ .

Lemma 1 ([55]). *Let $\mu \in \mathcal{P}(n)$ be a fixed set partition. For any function $E: \mathcal{P}(n)_{\geq \mu} \rightarrow \mathbb{C}$, there exists a unique $C: \mathcal{P}(n)_{\geq \mu} \rightarrow \mathbb{C}$ such that for all $\pi \in \mathcal{P}(n)$,*

$$E(\pi) = \sum_{\mu \leq \nu \leq \pi} C(\nu). \quad (28)$$

Notation. If (Y_1, \dots, Y_n) are n variables on the same probability space, with all their joint moments of degree less than n and for any $\pi \in \mathcal{P}(n)$, $\mathbb{E}_\pi(Y_1, \dots, Y_n) = \prod_{B \in \pi} \mathbb{E} \prod_{k \in B} Y_k$, then the value at a partition $\pi \in \mathcal{P}(n)$ of the unique solution to (28), is denoted by $C_{\mu, \pi}(Y_1, \dots, Y_n)$. It is a *relative cumulant*: for any $n \geq 1$, $C_{0_n, 1_n}(Y_1, \dots, Y_n)$, is the cumulant $C_n(Y_1, \dots, Y_n)$, whereas for any $\mu, \pi \in \mathcal{P}(n)$, with $\mu \leq \pi$,

$$C_{\mu, \pi}(Y_1, \dots, Y_n) = \prod_{S \in \pi} C_{\#\{B \in \mu: B \subset S\}} \left(\prod_{k \in B} Y_k, B \in \mu \text{ with } B \subset S \right).$$

For any pair of transpositions $\tau_1 = (a_1 b_1)$, $\tau_2 = (a_2 b_2)$, with $a_i < b_i$, let us write $\tau_1 \leq \tau_2$ when $b_1 \leq b_2$. \mathcal{W}_r^\uparrow is the set of tuples of transpositions (τ_1, \dots, τ_r) with $\tau_1 \leq \tau_2 \leq \dots \leq \tau_r$. For any partition $\pi \in \mathcal{P}(n)$, let us denote by S_π the subgroup of S_n consisting of permutations $\sigma \in S_n$ with $\sigma(B) = B$ for all blocks $B \in \pi$, set $\mathcal{W}_r^\uparrow(\pi) = \mathcal{W}_r^\uparrow \cap S_\pi^r$ and for any $A \subset [n]$, S_A , the group of permutations of A .

In the proof we will use the following standard fact on symmetric functions.

Lemma 2. For each integer $n \in \mathbb{N}$, and indeterminates t, x_1, x_2, \dots, x_n ,

$$\prod_{i \geq 1} (1 + x_i t) = \sum_{r \geq 0} e_r(x) t^r, \quad \prod_{i \geq 1} (1 - x_i t)^{-1} = \sum_{r \geq 0} h_r(x) t^r,$$

denoting respectively by $e_r(x) = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \dots x_{i_r}$ and $h_r(x) = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \dots x_{i_r}$ the elementary and complete symmetric functions.

Proof of Theorem 2. Let $\mu = (\mu_1, \dots, \mu_l) \vdash n$ and $\alpha \in S_n$ be a permutation of type μ and let $\pi_\alpha \in \mathcal{P}(n)$ be the set partitions with blocks given by cycles of α . Multilinearity of cumulants yields

$$C_l(\text{Tr} W^{-\mu_1}, \dots, \text{Tr} W^{-\mu_\ell}) = \sum_{\substack{i, j \in [N]^n: \\ i \circ \alpha = j}} C_{\pi_\alpha, 1_n}(W_{i(1)j(1)}^{-1}, \dots, W_{i(n)j(n)}^{-1}). \quad (29)$$

According to Proposition 4, if $(c-1)N > n$,

$$\begin{aligned} (-N)^n \Omega_{n, (1-c)N}^{-1} &= \prod_{i=1}^n (c-1 - N^{-1} J_i)^{-1} = (c-1)^{-n} \sum_{r \geq 0} h_r(J) ((c-1)N)^{-r} \\ &= (c-1)^{-n} \sum_{r \geq 0} ((c-1)N)^{-r} \sum_{(\tau_i)_{i=1}^r \in \mathcal{W}_r^\uparrow} \tau_1 \tau_2 \dots \tau_r, \quad (30) \end{aligned}$$

where we used Lemma 2 and the fact that the transpositions in J_i are all majorized by the transpositions in J_j when $i < j$. Combined with Proposition 3, this leads for any $i, j \in [N]^n$ to

$$\mathbb{E} W_{i(1)j(1)}^{-1} \cdots W_{i(n)j(n)}^{-1} = (c-1)^{-n} \sum_{r \geq 0} ((c-1)N)^{-r} \#\{(\tau_i) \in \mathcal{W}_r^\uparrow : j \circ \tau_1 \dots \tau_r = i\}.$$

On the one hand, after relabelling, the same argument applied to each block of a partition $\pi \in \mathcal{P}(n)$ gives

$$\mathbb{E}_\pi \left(W_{i(1)j(1)}^{-1}, \dots, W_{i(n)j(n)}^{-1} \right) = \prod_{B \in \pi} \left((c-1)^{-\#B} \sum_{r \geq 0} ((c-1)N)^{-r} \#\{(\tau_i) \in \mathcal{W}_r^\uparrow(B) : j|_B \circ \tau_1 \dots \tau_r = i|_B\} \right).$$

Distributing the terms in the product reads

$$(c-1)^{-n} \sum_{(r_B)_{B \in \pi} \in \mathbb{N}_+^\pi} \prod_{B \in \pi} ((c-1)N)^{-r_B} \#\{(\tau_i) \in \mathcal{W}_{r_B}^\uparrow(B) : j|_B \circ \tau_1 \dots \tau_r = i|_B\}.$$

Now, for any $(r_B)_{B \in \pi} \in \mathbb{N}_+^\pi$, because of the condition of monotonicity, for any collection $(w_B)_{B \in \pi} \in \prod_{B \in \pi} \mathcal{W}_{r_B}^\uparrow(B)$, there is a unique element of $\mathcal{W}_r^\uparrow(\pi)$ whose restrictions to blocks of π is given by w , where $r = \sum_{B \in \pi} r_B$. Hence, $\mathcal{W}_r^\uparrow(\pi)$ is in bijection with $\sqcup_{(r_B)_{B \in \pi} \in \mathbb{N}_+^\pi : r = \sum_{B \in \pi} r_B} \mathcal{W}_{r_B}^\uparrow(B)$. It follows that the latter expression reads

$$(c-1)^{-n} \sum_{r \geq 0} ((c-1)N)^{-r} \#\{(\tau_i) \in \mathcal{W}_r^\uparrow(\pi) : j \circ \tau_1 \dots \tau_r = i\}. \quad (31)$$

On the other hand, for any tuple $\mathcal{C} = (\sigma_1, \dots, \sigma_k) \in S_n^k$, let $\pi_{\mathcal{C}} \in \mathcal{P}(n)$ be the set partition given by the orbits of the group $\langle \sigma_1, \dots, \sigma_k \rangle$ and set for any $\pi \geq \nu \geq \pi_{\mathcal{C}}, r \geq 1$,

$$\mathcal{W}_r^\uparrow(\nu, \pi) = \{(\tau_i)_i \in \mathcal{W}_r^\uparrow(\pi) : \pi_{\mathcal{C}, \tau_1, \dots, \tau_r} = \nu\}.$$

Then, (31) implies that for any $\pi \in \mathcal{P}(n)_{\geq \pi_{\mathcal{C}}}$,

$$\mathbb{E}_\pi \left(W_{i(1)j(1)}^{-1}, \dots, W_{i(n)j(n)}^{-1} \right) = \sum_{\pi_{\mathcal{C}} \leq \nu \leq \pi} (c-1)^{-n} \sum_{r \geq 0} ((c-1)N)^{-r} \#\{(\tau_i) \in \mathcal{W}_r^\uparrow(\nu, \pi) : j \circ \tau_1 \dots \tau_r = i\}.$$

Using Lemma 1, it follows that for all $i, j \in [N]^n$ and $\nu \geq \pi_\alpha$,

$$C_{\nu, 1_n}(W_{i(1)j(1)}^{-1}, \dots, W_{i(n)j(n)}^{-1}) = (c-1)^{-n} \sum_{r \geq 0} ((c-1)N)^{-r} \#\{(\tau_i) \in \mathcal{W}_r^\uparrow(\nu, 1_n) : j \circ \tau_1 \dots \tau_r = i\}. \quad (32)$$

With this equation, we can now look back at (29) and write

$$C_\ell(\text{Tr} W^{-\mu_1}, \dots, \text{Tr} W^{-\mu_\ell}) = (c-1)^{-n} \sum_{r \geq 0} \sum_{\substack{i, j \in [N]^n : \\ i \circ \alpha = j}} ((c-1)N)^{-r} \#\{(\tau_i) \in \mathcal{W}_r^\uparrow(\pi_\alpha, 1_n) : j \circ \tau_1 \dots \tau_r = i\}.$$

For any $\beta \in S_\pi$ and $r \geq 1$, let us consider

$$\mathcal{W}_r^\uparrow(\pi_\alpha, \pi, \beta) = \{(\tau_i)_{i=1}^r \in \mathcal{W}_r^\uparrow(\pi_\alpha, \pi) : \alpha \tau_1 \dots \tau_r = \beta\}.$$

Fixing $r \geq 1$ in the last sum, the coefficient of $(c-1)^{-n-r}$ is

$$\begin{aligned} & N^{-r} \sum_{\substack{\beta \in S_n, \\ (\tau_i)_{i=1}^r \in \mathcal{W}_r^\uparrow(\pi_\alpha, 1_n, \beta)}} \#\{i, j \in [N]^n : j \circ (\alpha^{-1}\beta) = i, i \circ \alpha = j\} \\ &= \sum_{\substack{\beta \in S_n, \\ (\tau_i)_{i=1}^r \in \mathcal{W}_r^\uparrow(\pi_\alpha, 1_n, \beta)}} N^{\#\beta - r}. \end{aligned}$$

Now according to Riemann-Hurwitz formula [37, Remark 1.2.21], for any $\beta \in S_n$, $(\tau_i)_{i=1}^r \in \mathcal{W}_r^\uparrow(\pi_\alpha, 1_n, \beta)$, $\#\alpha + \#\beta - r = 2 - 2d$, for some $d \in \mathbb{N}$. Therefore, the last right-hand-side is

$$\sum_{r \geq 0, d \geq 0} N^{2-2d-\#\alpha} \#\mathcal{F}_{n,r,d}^\uparrow.$$

The first claim (12) follows by inspection.

The second claim (13) follows from the very same argument if, instead of (30), we start from the expression

$$\begin{aligned} (N)^{-n} \Omega_{n, cN} &= \prod_{i=1}^n (c + N^{-1} J_i) = c^n \sum_{r \geq 0} e_r(J) (cN)^{-r} \\ &= c^n \sum_{r \geq 0} (cN)^{-r} \sum_{(\tau_i)_{i=1}^r \in \mathcal{W}_r^{\uparrow\uparrow}} \tau_1 \tau_2 \dots \tau_r, \quad (33) \end{aligned}$$

where $\mathcal{W}_r^{\uparrow\uparrow}$ is the set of strictly monotone tuples of transpositions (τ_1, \dots, τ_r) , $\tau_1 < \tau_2 < \dots < \tau_r$. The proof of formula (13) proceeds mutatis mutandis

with \mathcal{W}_r^\uparrow replaced by $\mathcal{W}_r^{\uparrow\uparrow}$. The details of the calculations are left to the Reader. \square

We can now easily conclude.

Proof of Theorem 1 and Proposition 1. Unfolding the definitions of these statements and of Theorem 2, we get that for any $\mu \vdash n, r, g \geq 0$,

$$\sum_{\alpha \in S_n: [\alpha] = \mu} \# \mathcal{F}_{n,r,d}^\uparrow(\alpha) = \sum H_d^\uparrow(\mu, \nu)$$

and

$$\sum_{\alpha \in S_n: [\alpha] = \mu} \# \mathcal{F}_{n,r,d}^{\uparrow\uparrow}(\alpha) = \sum H_d^{\uparrow\uparrow}(\mu, \nu),$$

where in the right-hand-sides, we sum over $\nu \vdash n$ with $\#\nu = \#\mu + r - 2d$. The claims of Theorem 1 follow by inspection.

To prove Proposition 1, let us recall that any permutation $\sigma \in S_n$ can be uniquely factorized as $\sigma = \tau_1 \dots \tau_{|\sigma|}$ where $(\tau_1, \dots, \tau_{|\sigma|})$ is a strictly monotone tuple and $|\sigma| = n - \#\sigma$. Moreover, for any $\alpha \in S_n$, (α, σ) acts transitively on $[n]$ if and only if $(\alpha, \tau_1, \dots, \tau_{|\sigma|})$ does. Hence considering for any constellation $(\alpha, \beta, (\alpha\beta)^{-1})$, the unique tuple $(\tau_1, \dots, \tau_{|\beta|})$ with $\tau_1 < \tau_2 < \dots < \tau_{|\beta|}$ such that $\tau_1 \dots \tau_{|\beta|} = \beta$ leads to $\mathcal{C}_g(\mu, \nu) = H_g^{\uparrow\uparrow}(\mu, \nu)$, for any $\nu, \mu \vdash n$ and $g \geq 0$. \square

Remark 3. Proposition 1 can be proved more directly along the lines of the proof of Theorem 2 starting from the expression $(N)^{-n} \Omega_{n,cN} = \sum_{\beta \in S_n} N^{\#\beta - n} c^{\#\beta}$, without factorizing into transpositions.

Remark 4. Let us emphasize that in the proof of Theorem 2, the monotonicity condition was crucial for a factorisation property of the set of partitioned monotone paths to get (31).

Remark 5. Proposition 3 can be read as an equality of tensors in $\text{End}((\mathbb{C}^N)^{\otimes n})$. The left-hand-side commutes with the diagonal action of unitary matrices, whereas the right-hand-side can be viewed as the endomorphism given by the linear combination of permutations of tensors. (As already mentioned, this is an instance of **Schur-Weyl duality**.) It would have been more elegant but less elementary to write the above proof in this language.

2.2. A combinatorial proof of a duality formula. The two formulae in Theorem 1 have a striking similarity that we shall use to deduce Corollary 2. Therefor, we shall use the following decompositions of monotone minimal factorisations of a full cycle.

Denoting by T_n the set of all transpositions of S_n , we consider¹ for $r \geq 0$,

$$\mathcal{F}_{n,r}^\uparrow = \{(\tau_1, \dots, \tau_r) \in T_n^r : \#(1\ 2 \dots n)\tau_1 \dots \tau_r = r + 1, \tau_1 \leq \tau_2 \leq \dots \leq \tau_r\}$$

and

$$\mathcal{F}_{n,r}^{\uparrow\uparrow} = \{(\tau_1, \dots, \tau_r) \in \mathcal{F}_r^\uparrow : \tau_1 < \tau_2 < \dots < \tau_r\},$$

where by convention the empty sequence is the only element of $\mathcal{F}_{n,0}^\uparrow = \mathcal{F}_{n,0}^{\uparrow\uparrow} = \{()\}$. We wish to relate the sets $\mathcal{F}_n^\uparrow = \cup_{r \geq 1} \mathcal{F}_{n,r}^\uparrow$ and $\mathcal{F}_n^{\uparrow\uparrow} = \cup_{r \geq 0} \mathcal{F}_{n,r}^{\uparrow\uparrow}$. Let us define a map

$$\Phi_n : \mathcal{F}_{n+1}^\uparrow \longrightarrow \mathcal{F}_n^{\uparrow\uparrow}$$

setting for all $w = (\tau_1, \dots, \tau_r) \in \mathcal{F}_{n+1,r}^\uparrow$ given by $((a_1\ b_1), \dots, (a_r\ b_r))$, with $a_i < b_i$ for all i ,

$$\Phi(w) = (\tau_{i_1}, \dots, \tau_{i_l}),$$

where (i_1, \dots, i_l) are the record times of the sequence (b_1, \dots, b_r) , before reaching $n + 1$, defined inductively as follows. If $b_1 = n + 1$, $l = 0$ and $\Phi_n(w) = ()$. If $b_1 \leq n$, $i_1 = 1$ and $i_{m+1} = \inf\{t > i_m : b_t > b_{i_m}\}$ as long as $b_{i_{m+1}} \leq n$, while we set $l = m$ when $b_{i_{m+1}} > n$. For instance, $\Phi_4((1\ 3)(2\ 3)(1\ 5)(4\ 5)) = ((1\ 3))$. The main observation to prove the duality of Corollary 2 can be stated as follows.

Lemma 3. *For any $l \geq 0$, $w \in \mathcal{F}_{n,l}^{\uparrow\uparrow}$ and $r \geq l$,*

$$\#\Phi_n^{-1}(w) \cap \mathcal{F}_{n+1,r}^\uparrow = \binom{n-l}{r-l}.$$

Proof. Let us recall that for any permutation $\sigma \in S_n$ and any transposition (ab) , $\#\sigma.(ab) - \#\sigma$ is either 1, when a and b are in the same orbit of σ , or -1 otherwise. From this geometric fact follow two observations. When $(\tau_1, \dots, \tau_r) \in \mathcal{F}_{n,r}^\uparrow$,

1. for all $m \leq r$, $\#(1\ 2 \dots n)\tau_1 \dots \tau_m = m + 1$;
2. for all $m \leq r - 1$, writing $\tau_m = (ab)$ and $\tau_{m+1} = (cd)$, with $a < b$ and $c < d$, then

¹ We borrow here some notations from [41] but do not develop the relation with parking functions which would deserve further consideration.

- whether $[c, d] \supset [a, b]$,
- or $d = b$ and $c > a$.

Hence, any sequence $(\tau_1, \dots, \tau_r) \in \mathcal{F}_{n+1, r}$ can be written uniquely as

$$(a_1 b_1), (a_2 b_1), \dots, (a_{i_2-1} b_1), (a_{i_2} b_2), \dots, (a_{i_3-1} b_2), \dots \\ \dots, (a_{i_l} b_l), \dots, (a_{i_{l+1}-1} b_l), (a_{i_{l+1}} n+1), \dots, (a_r n+1), \quad (34)$$

where $1 \leq l \leq r$, $b_1 < b_2 < \dots < b_l$, $1 = i_1 < i_2 < \dots < i_l < i_{l+1} \leq r+1$ and for any $m \in [l+1]$,

$$a_{i_m} < a_{i_m+1} < \dots < a_{i_{m+1}-1} \text{ with } \{a_{i_m}, \dots, a_{i_{m+1}-1}\} \cap (a_{i_j} b_j] = \emptyset,$$

for all $j < m$, or as

$$(a_1 n+1), (a_2 n+1), \dots, (a_r n+1), \quad (35)$$

with $1 \leq a_1 < a_2 < \dots < a_r \leq n$. When $i_{l+1} = r+1$, by convention, no transposition acts on $n+1$. As illustrated in Figure 1, it follows that for any $0 \leq l < n$ and $w = ((x_1 b_1), \dots, (x_l b_l)) \in \mathcal{F}_{n,l}^{\uparrow\uparrow}$, with $x_i < b_i$ for all $i \in [l]$, the map

$$\Psi : \Phi_n^{-1}(w) \longrightarrow \{S \in \mathcal{P}([n]) : S \subset [n] \setminus \{b_1, \dots, b_l\}\}$$

that maps a sequence decomposed as in (34) or (35) to $\{a_1, \dots, a_r\} \setminus \{a_{i_1}, \dots, a_{i_l}\}$ and resp. $\{a_1, \dots, a_r\}$ when $l = 0$, is a bijection such that $\Psi(\mathcal{F}_{n+1, r}^{\uparrow} \cap \Phi_n^{-1}(w)) = \{S \in \mathcal{P}([n] \setminus \{b_1, \dots, b_l\}) : \#S = r-l\}$. The claim follows. \square

Proof of Corollary 2. Thanks to Theorem 2, applied to $\sigma = (1 \dots n+1)$,

$$\lim_{N \rightarrow \infty} (c-1)^{2n+1} \mathbb{E} \text{tr} W^{-n-1} = \sum_{r=0}^n (c-1)^{n-r} \# \mathcal{F}_{n+1, r}^{\uparrow} \quad (36)$$

and applied to $\sigma = (1 \dots n)$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \text{tr} W^n = \sum_{r=0}^{n-1} c^{n-r} \# \mathcal{F}_{n, r}^{\uparrow\uparrow}. \quad (37)$$

But applying Lemma 3 gives

$$\begin{aligned} \sum_{r=0}^n (c-1)^{n-r} \# \mathcal{F}_{n+1, r}^{\uparrow} &= \sum_{l=0}^n \sum_{w \in \mathcal{F}_{n, l}^{\uparrow\uparrow}} \sum_{r=l}^n (c-1)^{n-r} \# \Phi_n^{-1}(w) \cap \mathcal{F}_{n+1, r}^{\uparrow} \\ &= \sum_{l=0}^n \sum_{w \in \mathcal{F}_{n, l}^{\uparrow\uparrow}} \sum_{r=l}^n (c-1)^{n-r} \binom{n-l}{r-l} = \sum_{l=0}^{n-1} c^{n-l} \# \mathcal{F}_{n, l}^{\uparrow\uparrow}. \end{aligned}$$

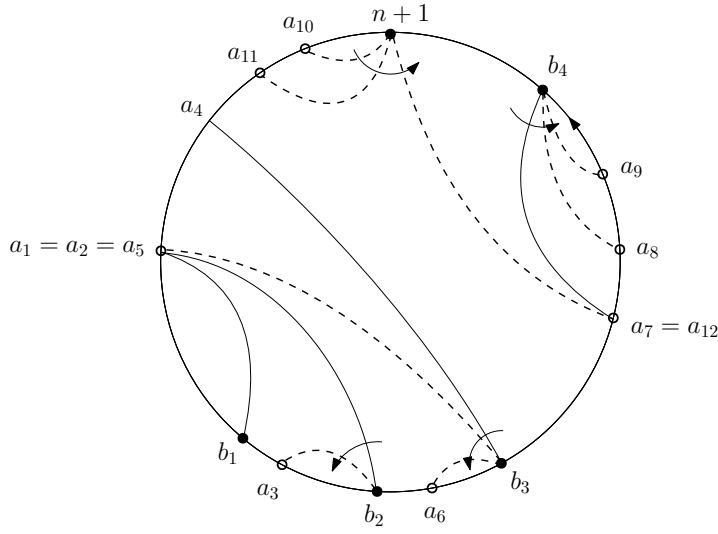


Figure 1. Representation of the decomposition of an element $w \in \mathcal{F}_{n+1,12}^\dagger$, where each transposition is represented by a strand that is dotted when it does not belong to $\Phi_n(w) \in \mathcal{F}_{n,4}^{\dagger\dagger}$. The set of white dots is $\Psi(w)$. The order of composition of the transpositions knowing only $\Psi(w)$ and the set of black dots is given first by the counter-clockwise order of the black dots and then by the counter-clockwise order of white dots around each black dot.

□

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Fabio Deelan Cunden, School of Mathematics and Statistics, University College Dublin, Dublin 4, Ireland

e-mail: fabio.cunden@ucd.ie

Antoine Dahlqvist, School of Mathematical and Physical Sciences, University of Sussex, Brighton, BN1 9RH, United Kingdom

e-mail: a.dahlqvist@sussex.ac.uk

Neil O’Connell, School of Mathematics and Statistics, University College Dublin, Dublin 4, Ireland

e-mail: neil.oconnell@ucd.ie